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THE DEGREE OF THE DIVISOR OF JUMPING RATIONAL CURVES

Z. Ran
Math Dept. UCR, Riverside CA 92521 USA
ziv@ math.ucr.edu

Abstract

For a semistable reflexive sheaf E of rank r and $c_1 = a$ on $\mathbb{P}^n$ and an integer d such that $r|ad$, we give sufficient conditions so that the restriction of E on a generic rational curve of degree d is balanced, i.e. a twist of the trivial bundle (for instance, if E has balanced restriction on a generic line, or $r = 2$ or E is an exterior power of the tangent bundle). Assuming this, we give a formula for the 'virtual degree', interpreted enumeratively, of the (codimension-1) locus of rational curves of degree d on which the restriction of E is not balanced, generalizing a classical formula due to Barth for the degree of the divisor of jumping lines of a semistable rank-2 bundle. This amounts to computing a certain determinant line bundle associated to E on a parameter space for rational curves, and is closely related to the 'quantum K-theory' of projective space.

INTRODUCTION

Let E be a reflexive sheaf (or vector bundle) of rank r on a projective space $\mathbb{P}^n, n \geq 2$. By a theorem of Grothendieck, the pullback E_C of E to the nonsingular model of any rational curve in $\mathbb{P}^n$ can be decomposed as

$$E_C = \bigoplus_{i=1}^{r} \mathcal{O}(k_i), \quad k_1 \geq \ldots \geq k_r,$$

where the sequence $(k_1, \ldots, k_r)$ is uniquely determined and called the *splitting type* of E on C. By semi-continuity, splitting type determines a natural locally closed stratification of any parameter space for rational curves, which is an important aspect of the geometry of E.

For E semistable of rank $r = 2$, a well-known theorem of Grauert-Mülich [OSS] says that if $r = 2$, the splitting type of E on a generic line $L \subset \mathbb{P}^n$ is either (k, k) ($c_1(E)$ even) or $(k, k-1)$ ($c_1(E)$ odd). One of the most important geometric objects associated to E of rank 2 is the locus of *jumping lines*, i.e. the locus of lines L such that E_L is not of the above generic form. This locus is especially important in case $c_1(E)$ is even, when it is actually a divisor. A well-known formula due originally to Barth computes the degree of the divisor of jumping lines (with a suitable scheme structure).

The purpose of this paper is to generalize Barth's formula to the case of bundles of arbitrary rank r and rational curves of any degree d such that

$$r|dc_1(E), \tag{0.1}$$

essentially under the condition that the restriction of E on a generic rational curve C of degree d is balanced, i.e. has splitting type (k^r), so that the locus of rational

curves C for which E_C is unbalanced is of pure codimension 1 and may be called the *divisor of jumping rational curves* of degree d of E.

Now the balancedness condition is not satisfied for all semistable sheaves E even for lines. However, we will introduce below a condition (condition AB), which states that the restriction of E on a generic line L is 'almost balanced', i.e. has splitting type $(k^s, (k-1)^{r-s})$ for some k, s, (so if $r|c_1(E)$ then E_L is actually balanced). Assuming this, we will then introduce a certain transversality condition T and show that with this condition the restriction E_C to a generic rational curve of any degree d is almost balanced, so that under (0.1) a divisor of jumping rational curves of degree d may be defined. We will also show that conditions AB and T are satisfied whenever either $r|c_1(E)$ and AB holds, or $r = 2$ or E is an exterior power of the tangent bundle.

Our main result (Theorem 3.1), which assumes conditions AB and T, computes the 'degree' of the divisor of jumping rational curves, which is interpreted as the weighted number of these curves incident to a generic collection of linear spaces (assuming of course that the total codimension of the incidence conditions equals the dimension of the divisor). More precisely, our formula expresses this degree in terms of some other enumerative invariants which have been computed before, e.g. in [P], [R1],[R2],[R3].

Note that even a homogeneous bundle like the tangent bundle in general admits jumping rational curves of degree $d > n$. In fact, the divisors of jumping rational curves associated to homogeneous vector bundles are an interesting class of projectively invariant divisors on any parameter space of rational curves (be it the Chow variety considered here, or Kontsevich's space or whatever).

This note was motivated by a talk by Givental [G] on his 'quantum K-theory', still under construction. This theory seeks to compute expressions of the form

$$\chi(M_{0,m}(\mathbb{P}^n, d), ev^*(E_1 \boxtimes \ldots \boxtimes E_m)),$$

where $M_{0,m}(\mathbb{P}^n, d)$ is Kontsveich's space of stable m-pointed rational curves of degree d and $ev : M_{0,m}(\mathbb{P}^n, d) \to (\mathbb{P}^n)^m$ is the evaluation map. As will emerge below, our formula essentially amounts to a computation of

$$\chi(M_{0,1}(\mathbb{P}^n, d)_{(A.)}, ev^*(E)),$$

where $M_{0,1}(\mathbb{P}^n, d)_{(A.)}$ is the normalization of a 1-dimensional subvariety of $M_{0,1}(\mathbb{P}^n, d)$ defined by incidence to a generic collection $(A.)$ of linear spaces. By the Riemann-Roch formula, the latter may also be identified as an intersection number in $M_{0,m}(\mathbb{P}^n, d)$; for instance in case all the A_i are points, hence have trivial normal bundle, it equals

$$\chi(M_{0,m}(\mathbb{P}^n, d), ev_1^*(E), ev_2^*(h^n) \cup \ldots \cup ev_m^*(h^n)),$$

where h is the hyperplane class, and in general the 'relative' Euler characteristic may be defined by

$$\chi(X, E, b) = \int_X \mathrm{ch}(E) \cup \mathrm{td}(T_X) \cup b \quad ,$$

and coincides with the Euler characteristic of the restriction of E on any smooth subvariety B with fundamental class b and trivial normal bundle (which may or may not exist); in our case $B = (ev_2 \times \ldots \times ev_m)^{-1}(pt.)$ has trivial normal bundle.

Though our formula is apparently new and independent of any quantum methods, it might in principle become accessible by Givental's methods at some point. Indeed these methods, unlike ours, might yield results when the jumping locus has codimension > 1.

The paper is organised as follows. In Sect.1 we study the Chow compactification of the family of rational space curves. It does not seem to be generally known that this compactification is well-behaved at least in codimension 1 (and perhaps in codimension 2 as well, as long as multiple components don't appear), so we have given a self-contained treatment here. The results come out as expected. In Sect.2 we study qualitatively the restriction of a bundle on a projective space to rational curves, focusing on criteria to ensure that the restriction on a generic rational curve is balanced. Our enumerative formula is given in Sect.3. The proof is based, not surprisingly, on the Riemann-Roch formula.

1. RATIONAL CURVES

Here we review some qualitative results about families of rational curves in $\mathbb{P}^n$. See also [R1][R2][R3] and references therein for details and proofs. In what follows we fix $n \geq 2$ and denote by $\bar{V}_d$ or $\bar{V}_{d,n}$ the closure in the Chow variety of the locus of irreducible nodal (if $n = 2$) or nonsingular (if $n > 2$) rational curves of degree d in $\mathbb{P}^n$, with the scheme structure as closure, i.e. the reduced structure (recall that the Chow form of a reduced 1-cycle Z is just the hypersurface in $G(n-2, \mathbb{P}^n)$ consisting of all linear spaces meeting Z). Thus $\bar{V}_d$ is irreducible reduced of dimension $(n+1)d+n-3$. Let $A_1, \dots, A_k$ be a generic collection of linear subspaces of respective codimensions $a_1, \dots, a_k, 2 \leq a_i \leq n$ in $\mathbb{P}^n$. We denote by

$$B = B_d = B_d(a.) = B_d(A.)$$

the normalization of the locus (with reduced structure)

$$\{(C, P_1, \dots, P_k) \ : \ C \in \bar{V}_d, P_i \in C \cap A_i, i = 1, \dots, k\}$$

which is also the normalization of its projection to $\bar{V}_d$, i.e. the locus of degree-d rational curves (and their specializations) meeting $A_1, \dots, A_k$. We have

$$\dim B = (n+1)d + (n-3) - \sum (a_i - 1). \tag{1.1}$$

When $\dim B = 0$ we set

$$N_d(a.) = \deg(B). \tag{1.2}$$

When $n = 2$, all $a_i = 2$ so the a's may be suppressed. For $n > 2$, k is called the *length* of the condition-vector $(a.)$. The numbers $N_d(a.)$, first computed in general by Kontsevich and Manin, are computed in [R2],[R3] by an elementary method based on recursion on d and k.

Now suppose $\dim B = 1$ and let

$$\pi : X \to B \tag{1.3}$$

be the normalization of the tautological family of rational curves, and $f : X \to \mathbb{P}^n$ the natural map. The following summarizes results from [R2][R3] (proved mostly in the references therein):

Theorem 1.1. *(i) X is smooth .*

(ii) Each fibre C of π is either

(a) a $\mathbb{P}^1$ on which f is either an immersion with at most one exception which maps to a cusp ($n = 2$) or an embedding ($n > 2$); or

(b) a pair of $\mathbb{P}^1$'s meeting transversely once, on which f is an immersion with nodal image ($n = 2$) or an embedding ($n > 2$); or

(c) if $n = 3$, a $\mathbb{P}^1$ on which f is a degree-1 immersion such that $f(\mathbb{P}^1)$ has a unique singular point which is an ordinary node.

(iii) If $n > 2$ then $\bar{V}_{d,n}$ is smooth along the image $\bar{B}$ of B, and $\bar{B}$ is smooth except, in case some $a_i = 2$, for ordinary nodes corresponding to curves meeting some A_i of codimension 2 twice. If $n = 2$ then $\bar{V}_{d,n}$ is smooth in codimension 1 except for a cusp along the cuspidal locus and normal crossings along the reducible locus, and $\bar{B}$ has the singularities induced from $\bar{V}_{d,n}$ plus ordinary nodes corresponding to curves with a node at some A_i, and no other singularities.

The basic idea is that one can first get a handle on what curves occur in $\bar{B}$ by the standard technique of semistable reduction (actually, normalization is sufficient) plus dimension counting (as, e.g. in Harris' work on the Severi problem); then doing the deformation theory for the curves which do occur is easy enough. For the convenience of the reader we will give a complete proof for $n > 2$.

Let H_d denote the (scheme-theoretic) closure in the Hilbert of the family of nonsingular rational curves of degree d, and $H_d^0 \subset H_d$ be the open subset of reduced curves with normal crossings, which is well known to be smooth (see also below). Let

$$\pi_d : X_d \to \bar{V}_d, \quad \pi_{H_d} : X_{H_d} \to H_d$$

be the universal cycle (resp. universal curve). There is a natural morphism (cf. [K])

$$c : H_d \to \bar{V}_d$$

which assigns to a curve C its Chow form, which is the divisor on the Grassmannian $G(n-2, \mathbb{P}^n)$ consisting of subspaces meeting C, with multiplicities if C is not generically reduced. Thus $\bar{V}_d$ is naturally embedded in a projective space parametrizing a suitable linear system on the Grassmannian. Clearly c is one to one on the subset $H_d^1 \subset H_d$ consisting of reduced curves. In fact, more is true:

Lemma 1.2. *c is unramified at any reduced subscheme.*

proof. Let C be reduced and pick a nonzero $v \in T_{[C]}H_d$. Let $p \in C$ be a general point where $v_p \neq 0$, and let L be a general $(n-2)-$ plane through p, which is also a general point in some component of $c(C)$. Let $\tilde{L}$ be the lift of L to $\mathbb{C}^{n+1}$, and $\tilde{(p)} \subset \tilde{(L)}$ the 1-dimensional subspace lifting p. Then the tangent space to the Grassmannian at L may be identified with

$$\mathrm{Hom}(\tilde{L}, \mathbb{C}^{n+1}/\tilde{L}),$$

while the tangent space to the divisor $c(C)$ is

$$\{\phi \in \mathrm{Hom}(\tilde{L}, \mathbb{C}^{n+1}/\tilde{L}) : \phi(\tilde{p}) \equiv 0 \mod T_pC\}.$$

Choosing L general through p, we can arrange that

$$v_p \notin < T_pC, T_pL >,$$

and it follows that as C moves infinitesimally according to v, we can move L preserving incidence to C and going outside of $c(C)$, so $d_{[C]}(v) \neq 0$. Note that the Lemma and the proof are valid for pure-dimensional subschemes of any dimension. □

Now the basic codimension-1 dimension counting result for $\bar{V}_d$ is the following

Proposition 1.3. *Let $W \subset \bar{V}_d$ be any codimension-1 subvariety and $[C] \in W$ a general curve. Then C is either*

(i) a smooth embedded $\mathbb{P}^1$; or

(ii) a pair of smooth embedded $\mathbb{P}^1$'s meeting transversely at one point; or

(iii) only if $n = 3$, an irreducible immersed rational curve with one normal crossing.

Moreover $\bar{V}_d$ is smooth at $[C]$ in each case and has tangent space $H^0((I_C/I_C^2)^*)$ in case (i) or (ii), or $H^0(N_f)$, where $f : \mathbb{P}^1 \to C$ is the normalization, in cases (i) or (iii).

proof. We will use the following variant of Kleiman transversality:

Lemma 1.4. *Let $\{Z_s : s \in S\}$ be a family of k-cycles in $\mathbb{P}^n$ which is PGL_n-equivariant (i.e. S is PGL_n-invariant and $gZ_s = Z_{gs}$), and let $U \subset \mathbb{P}^n$ be a (purely) codimension-c subvariety, $c > k$. Then the locus $S_U := \{s \in S : Z_s \cap U \neq \emptyset\}$ is of codimension $c - k$ in S.*

proof. Take any $s \in S$ and any $(c-k-1)-$ dimensional subvariety $Q \subset PGL_n$. Kleiman transversality says that

$$\overline{(\bigcup_{g \in Q} gZ_s)} \cap g_0U = \emptyset$$

for general $g_0 \in PGL_n$, hence

$$\overline{(\bigcup_{g \in Q} g_0^{-1}gZ_s)} \cap U = \emptyset.$$

This easily implies that the intersection of S_U with the PGL_n-orbit of Z_s is of codimension $c - k$. Thus S_U meets every orbit in codimension $c - k$, and it follows easily that S_U is of codimension $c - k$. □

We now return to the proof of the Proposition. If W fails to be PGL_n-invariant, then its general element $[C]$ is general in $\bar{V}_d$, hence smooth. Hence we may assume W is PGL_n-invariant. By semistable reduction, there exists a family

$$Y \to T$$

with general fibre $\mathbb{P}^1$ and special fibre

$$Y_0 = \bigcup Y_{0,i}$$

with normal crossings, and with a surjective map $h : Y_0 \to C$. Set

$$h_i = h|_{Y_{0,i}}, h_{i*}[Y_{0,i}] = m_i C_i \text{ or } 0, d_i = \deg(C_i), k = \#\{i : m_i > 0\}.$$

We may assume $m_1 > 0$ and that C_1 is non-disconnecting (i.e. $\bigcup\limits_{i>1} C_i$ is connected). Then from Lemma 1.4 it follows that

$$(n+1)d + n - 4 = \dim W = \dim\{C\} \leq \dim\{C_1\} + \dim\{\bigcup_{i>1} C_i\} - n + 2$$

$$\leq (n+1)d_1 + n - 3 + \dim\{\bigcup_{i>1} C_i\} - n + 2 \leq \dots$$

$$\leq (n+1)\sum d_i + k(n-3) - (k-1)(n-2) = (n+1)\sum d_i + n - 3 - (k-1).$$

It follows at once that $\sum d_i = d$, so that all nonzero m_i are equal to 1, and that $k = 1$ or 2, and in the latter case all inequalities above are equalities.

Now suppose $k = 2$, so

$$C = C_1 \cup_p C_2.$$

Then (C_1, C_2) must be a general point in the locus of intersecting curves in $\bar{V}_{d_1} \times \bar{V}_{d_2}$, which has codimension $n-2$, and it follows easily that C_1 and C_2 are smooth; moreover as the stabiliser in PGL_n of a point $p \in \mathbb{P}^n$ acts transitively on $\mathbb{P}^n - p$ and on $T_p\mathbb{P}^n$, it follows easily that C_1 and C_2 meet transversely once. Now as C is a locally complete intersection, the tangent space to H_d at $[C]$ is given by

$$H^0(N),\ N = (I_C/I_C^2)^*.$$

On the other hand the tangent space to the locally trivial deformations is given by

$$H^0(N'),$$

where N' is the kernel of a map

$$N \to \mathbb{C}_p$$

to a skyscraper sheaf at p, which assigns to a deformation of C the corresponding deformation of the germ (C, p), which is just an ordinary node (cf. [S]). Note that N' may also be identified as the kernel of the natural map

$$N_{C_1} \oplus N_{C_2} \to T_p\mathbb{P}^n/(T_pC_1 + T_pC_2)\ ,$$

$$(v_1, v_2) \mapsto v_{1,p} - v_{2,p},$$

hence clearly $H^1(N') = 0$. It follows easily as in [S] that the germ of H_d at $[C]$ is smooth and maps unramifiedly to the deformation space of (C, p). Since the total space of the versal deformation of (C, p) is just given by $xy = t$, it is smooth, and it follows that the universal curve X_d is smooth along its fibre over $[C]$.

It remains to consider the case where C is irreducible of degree d, hence given by projecting a rational normal curve

$$C_d \subset \mathbb{P}^d$$

from a center

$$M = \mathbb{P}^{d-n-1},\ M \cap C_d = \emptyset.$$

Recall that an arbitrary length-k subscheme of C_d spans a $\mathbb{P}^{k-1}$, so if C is not embedded then for any (C_d, M) such that $C = \mathrm{proj}_M(C_d)$, M must meet some k-secant $\mathbb{P}^{k-1}$ to C_d in a $\mathbb{P}^{k-2}$, $k \geq 2$. By a straightforward dimension count, the family of pairs (C_d, M) with the latter property, for any fixed k, is of codimension

$$1 + (k-1)(n-2)$$

and, under the mapping

$$(C_d, M) \mapsto \mathrm{proj}_M(C_d)$$

maps to a subfamily of the same codimension in $\bar{V}_d$. Since $n > 2$, this codimension could equal 1 only if $n = 3$, $k = 2$, hence the set of C's which are not embedded has codimension 1 only if $n = 3$. We can see similarly that if $n = 3$ a general nonembedded C has one normal crossing and corresponds to M meeting the secant variety of C_d in one general point.

Now for any irreducible C of degree $d \geq 3$, pick general points

$$p_1, p_2, p_3 \in C$$

and transverse hyperplanes $H_i \ni p_i$, and let

$$f : \mathbb{P}^1 \to C$$

be the normalisation and set $p_i' = f^{-1}(p_i)$. Consider a space D parametrising deformations f' of f so that

$$f'(p_i') \in H_i.$$

Then clearly

$$T_f D = H^0(T')$$

where

$$T' \subset f^*(T_{\mathbb{P}^n})$$

is the (full-rank) subsheaf of vector fields tangent to H_i at p_i', $i = 1, 2, 3$. As

$$f^*(T_{\mathbb{P}^n})(-p_1' - p_2' - p_3') \subset T',$$

clearly $H^1(T') = 0$, hence D is smooth at f. Moreover the natural map

$$D \to \bar{V}_d$$

$$f' \mapsto f'(\mathbb{P}^1)$$

is clearly one to one and is unramified at f by an evident variant of Lemma 1.2. Therefore $\bar{V}_d$ is smooth at $[C]$ (which clearly implies that the normalization of the total space of the universal cycle, which has a smooth fibre over $[C]$, is itself smooth along this fibre). Also, it is clear that the natural map $T' \to N_f$ induced an isomorphism

$$H^0(T') \to H^0(N_f).$$

This completes the proof of Proposition 1.3. $\square$

We can now complete the proof of Theorem 1.1. Let

$$X'_d \to \bar{V}_d$$

be the normalization of the universal cycle. Then X'_d is irreducible of dimension $(n+1)d+n-2$ and nonsingular in codimension 1 . Likewise the fibred product

$$(X'_d)^k \to \bar{V}_d$$

is irreducible of dimension $(n+1)d+n-3+k$ and nonsingular in codimension 1 (its singularities come from singularities of $\bar{V}_d$ and repeated singular points of fibres). Moreover the fibres of $(X'_d)^k$ over $\bar{V}_d$ are obviously k-dimensional. Now consider the incidence variety

$$I = \{(C, p_1, ..., p_k, A_1, ..., A_k) : p_i \in A_i, i = 1, ...k\}$$
$$\subset (X'_d)^k \times G(n-a_1, \mathbb{P}^n) \times ... \times G(n-a_k, \mathbb{P}^n)\}.$$

This is obviously a fibre bundle over $(X'_d)^k$, hence irreducible of dimension

$$(n+1)d+n-3+k+\sum a_i(n-a_i),$$

i.e. codimension $\sum a_i$ in the product, and nonsingular in codimension 1. It follows that if

$$\sum(a_i-1) = (n+1)d+n-4$$

then a general fibre B of I over $G(n-a_1, \mathbb{P}^n) \times ... \times G(n-a_k, \mathbb{P}^n)$ is smooth and 1-dimensional and the image $\bar{B}$ of B in $\bar{V}_d$ can be made disjoint from any given codimension-2 subvariety, hence the curves in $\bar{B}$ are as claimed. The smoothness of X as in Theorem 1.1 can be proved similarly by considering a suitable incidence variety in

$$(X'_d)^{k+1} \times G(n-a_1, \mathbb{P}^n) \times ... \times G(n-a_k, \mathbb{P}^n)$$

(with incidence conditions on the first k points). Finally, as B is smooth, clearly singularities of $\bar{B}$ come from curves meeting some A_i more than once (or nontransversely), and the remaining assertions about these singularities can be proved by a dimension-counting argument on the rational normal curve similar to the one we did above. This completes the proof of Theorem 1.1. $\square$

It follows from Theorem 1.1 that we can speak about a 'general reducible boundary curve' of V_d as being a 1-nodal curve

$$C_1 \cup_p C_2$$

which is either embedded as such (if $n > 2$) or maps to a reducible nodal curve with one distinguished separating node (if $n = 2$).

Remark 1.5. Another fact which follows easily from the above discussion is that for any

$$b = (C, p_1, ..., p_k) \in B,$$

we can identify the tangent space $T_b B$ with

$$\{v \in H^0(N) : v_{p_i} \equiv 0 \mod T_{p_i} A_i, i = 1, ..., k\}$$

where N is either $(I_C/I_C^2)^*$ in case (a) or (b), or N_f in case (a) or (c). This implies, in the notation of [R3],Sect 2, that not only is $B = \bigcap B_i$ but

$$T_b B = \bigcap_i T_b B_i$$

as well, so that B is the complete transverse intersection of the B_i in B^+, a fact which was implicitly used in the computation o the genus of B in [R3].

2. JUMPING RATIONAL CURVES

Here we discuss some qualitative generalities about restriction of vector bundles from $\mathbb{P}^n$ to rational curves. See [OSS] for details on vector bundles over projective spaces.

A vector bundle E_C of rank r on a rational curve C is said to be *almost balanced* if it can be decomposed

$$E_C \simeq s\mathcal{O}(k) \oplus (r-s)\mathcal{O}(k-1). \tag{2.1}$$

In this case the subsheaf $s\mathcal{O}(k) \subseteq E_C$ is well-defined and determines a canonical 'positive' subspace $V_C(p) \subseteq E(p) := E \otimes \mathcal{O}_p$ for any $p \in C$.

Now let E be a semistable reflexive sheaf of rank r on $\mathbb{P}^n$ and chern class $c_1(E) = a \in \mathbb{Z}$. We introduce the following

Condition AB. *The restriction E_L of E on a general line is almost balanced.*

By the Grauert-Mülich Theorem, condition AB is satisfied if E is semistable of rank 2. Also, this condition is obviously satisfied whenever

$$E = \bigwedge^m T_{\mathbb{P}^n}$$

for any m (though it fails for $E = \mathrm{Sym}^m T_{\mathbb{P}^n}, m > 1$). Assuming this condition holds, we try to get a similar conclusion for rational curves of higher degree. To this end, consider the following

Transversality condition T. *Given a general point $p \in \mathbb{P}^n$, an arbitrary subspace $W \subset E(p)$, and a general line $L \ni p$, the positive subspce $V_L(p)$ is transverse to W.*

Lemma 2.1. *Assuming condition AB, condition T is satisfied provided either*
(i) $a \equiv 0 \mod r$*; or*
(ii)$r = 2$*; or*
(iii) $E = \bigwedge^m T_{\mathbb{P}^n}$.

proof. Case (i) is obvious since then $V_L(p) = E(p)$. In case (ii), if a is odd, it is well known [OSS] that by semistability of E the 1-dimensional subspace $V_L(p)$ varies with L, which is sufficient. In case (iii) $V_L(p)$ is the space of 'multiples' of $T_L(p)$ and again the result is clear.

Thus, conditions AB and T are both satisfied whenever either E has generic splitting type (k^r) on lines, or has rank 2 or is an exterior power of the tangent bundle.

Now we want to study restrictions of E either to general rational curves or to general reducible curves in $\bar{V}_{d,n}$. To this end we make another definition. Consider a 1-nodal curve

$$C = C_1 \cup_p C_2$$

with each $C_i \simeq \mathbb{P}^1$. A bundle E_C on C is said to be *almost balanced* if each E_{C_i} is almost balanced and the induced positive subspaces

$$V_1,\ V_2 \subseteq E(p)$$

are in general position. It is an easy execise that in this case we can write

$$E_C \simeq (\oplus L_i) \oplus (\oplus M_j) \tag{2.2}$$

where for some k, each L_i (resp. M_j) is a line bundle of total degree k (resp $k-1$). Moreover the 'positive' subsheaf $\oplus L_i \subseteq E_C$ is canonically defined and for any general point $q \in C_1$ or C_2 the corresponding subspace $V \subseteq E(q)$ can be identified in an evident sense with either $V_1 + V_2$ or $V_1 \cap V_2$. Note that the decomposition (2.2) implies easily that for any small deformation $(E_{C'}, C')$ of (E_C, C) where C' is a smooth $\mathbb{P}^1$, $E_{C'}$ is almost balanced.

Proposition 2.2. *Let E be a reflexive sheaf on $\mathbb{P}^n$ satisfying conditions AB and T, and let C be either a general element or a general reducible boundary element of $\bar{V}_d$. Then E_C is almost balanced.*

proof. We use induction on d, the case $d = 1$ being exactly condition AB. Assuming the assertion holds for $d-1$, specialise a general curve C of degree d to a general reducible $C_0 = C_1 \cup_p L$ with L a line. By property T, clearly E_{C_0} is almost balanced, hence so is E_C for a general C.

For a general reducible $C = C_1 \cup_p C_2$, we know E_{C_i} is almost balanced and moreover, as each C_i may be viewed as a specialisation of a polygon, it follows that the positive subspaces $V_i \subseteq E(p)$ may be assumed transverse, hence E_C is almost balanced. $\square$

Example 2.3. We consider the case of the tangent bundle $E = T_{\mathbb{P}^n}$. For any irreducible rational curve

$$C \to \mathbb{P}^n$$

of degree $d \equiv 0 \mod n$, given by a polynomial vector

$$(f_0, ..., f_n) \in \oplus H^0(\mathcal{O}_C(d)),$$

we have an exact sequence

$$0 \to \mathcal{O}_C \to \oplus \mathcal{O}_C(d) \to E_C \to 0 \ .$$

Dualising and twisting by d, we see that sections of $E_C^*(d+k)$ correspond to syzygies of degree k among the f_i, i.e. relations of the form

$$\sum g_i f_i = 0, \qquad g_i \in H^0(\mathcal{O}_C(k)).$$

It is immediate from this that C is a jumping curve iff it admits a syzygy of degree $\frac{d}{n} - 1$, while any curve admits a syzygy of degree $\frac{d}{n}$. It is also easy to see that C is a jumping curve iff the ideal generated by $f_0, ..., f_n$ in the homogeneous coordinate ring of C fails to contain $H^0(\mathcal{O}_C(d + \frac{d}{n} - 1))$, while this ideal never contains $H^0(\mathcal{O}_C(d + \frac{d}{n} - 2))$.

3. COUNTING JUMPING RATIONAL CURVES

We continue with the notations above and assume that E satisfies conditions AB and T. Then we may define the *jumping locus*

$$\mathcal{J}_{d,E} \subset \bar{V}_d,$$

as a set, to be the closure of the set of irreducible reduced C such that E_C is not almost balanced. Now if it happens that

$$-r < ad \leq 0, \tag{3.1}$$

then it is easy to endow $\mathcal{J}_{d,E}$ with a global scheme structure: namely let

$$\Pi : \mathcal{X} \to V_d$$

be the tautological family and

$$\mathcal{F} : \mathcal{X} \to \mathbb{P}^n$$

the natural map, and let $\mathcal{J}_{d,E}$ be the *Fitting subscheme*

$$Fit_1(R^1\Pi_*(\mathcal{F}^*E)),$$

defined by 1st fitting ideal of $R^1\Pi_*(\mathcal{F}^*E)$. This what is done in [OSS] for $d = 1$. For our purposes however, the hypothesis (3.1) is too restrictive. Without it one can still endow $\mathcal{J}_{d,E}$, at least in the event it had codimension 1, with a scheme structure 'slice by slice' , as we now proceed to do.

We now assume that

$$r|ad. \tag{3.2}$$

In view of Proposition 2.2 this implies that for a general $C \in V_d,$, E_C is in fact *balanced*. Let $\pi : X \to B, f : X \to \mathbb{P}^n$ be as in Sect. 1, and let

$$s_i \subset X, i = 1, ..., k \tag{3.3}$$

be the tautological section corresponding to A_i. Let $D = D_{t.}$ be any divisor of the form

$$D = \sum t_i s_i \tag{3.4}$$

such that $\sum t_i = \frac{ad}{r} + 1$. Set

$$G = f^*(E)(-D). \tag{3.5}$$

Thus for a general fibre $F_b = \pi^{-1}(b)$ we have

$$G_{F_b} = r\mathcal{O}(-1).$$

We define $\mathcal{J}_{d,E,B}$ to be the part of the first Fitting scheme $Fit_1(R^1\pi_*(G))$ supported in the open subset $B^0 \subseteq B$ corresponding to irreducible curves. It is easy to see that this is independent of the choice of twisting divisor D. In particular, taking $D = (ad + r)s_i$, our scheme structure coincides with the natural scheme structure on $\mathcal{J}_{d,E,B}$ which defined, at least through codimension 1 over the locus of curves in $\bar{V}_d$ incident to A_i, by virtue of the existence of a canonical section. Now set

$$J_{d,E}(a.) = c_1(\mathcal{J}_{d,E,B}). \tag{3.6}$$

This evidently depends only on the $(a.)$, and it is this number that we will compute.

To state our formula conveniently we introduce some objects from [R2][R3]. Set

$$m_i = m_i(a.) = -s_i^2, i = 1, ..., k. \tag{3.7}$$

Note that if $a_i = a_j$ then $m_i = m_j$. In particular for $n = 2$ they are all equal. It is shown in [R2][R3] that these numbers can all be computed recursively in terms of data of lower degree d and (for $n > 2$) lower length. For instance for $n = 2$ we have

$$2m_1 = \sum_{d_1+d_2=d} N_{d_1}N_{d_2}d_1d_2\binom{3d-4}{3d_1-2}. \tag{3.8}$$

Note that

$$s_i.s_j = N_d(..., a_i + a_j, ..., \hat{a}_j, ...), i \neq j, \tag{3.9}$$

so this number may be considered known. Hence D^2 may be considered known. Also, let R_i be the sum of all fibre components not meeting s_i . We have

$$R_i.s_j = m_i + m_j + 2s_i.s_j \tag{3.10}$$

so this number is computable as well (actually we showed in [R2] that this is computable in terms of lower-degree data, and the m_i were computed from that).

Next, set

$$L = f^*(\mathcal{O}(1)),$$

and note that

$$L^2 = N_d(2, a.),\ L.s_i = 0,\ i = 1, ..., k. \tag{3.11}$$

Also, it is easy to see as in [R3] that

$$L.R_i = \sum_{d_1+d_2=d} \binom{3d-1}{3d_1-1} d_1 d_2^2 N_{d_1} N_{d_2},\ n = 2$$

$$L.R_i = \sum d_2 N_{d_1}(a^1_., a_i, n_1) N_{d_2}(a^2_., n_2),\ n > 2, \tag{3.12}$$

the summation for $n > 2$ being over all $d_1 + d_2 = d, n_1 + n_2 = n$ and all decompositions

$$A. = (A_i) \coprod (A^1_.) \coprod (A^2_.)$$

(as unordered sequences or partitions).

Recall that the geometric genus $g = g(B)$ was computed in [P][R2] for $n = 2$ and in [R3] in general. Next, define for any index-set $I \subseteq \{1, ..., k\}$ of cardinality $|I|$

$$t_I = \sum_{i \in I} t_i, \tag{3.13}$$

$$h(a., t.) = \sum N_{d_1} N_{d_2} \binom{3d-1-|I|}{3d_1-1-|I|} (rt_I - ad_1 - r),\ n = 2$$

$$h(a., t.) = \sum N_{d_1}(a_i : i \in I) N_{d_2}(a_i : i \notin I)(rt_I - ad_1 - r),\ n > 2 \tag{3.14}$$

the sum being extended over all $d_1 + d_2 = d$ and all index-sets I such that $t_I > \lceil \frac{ad_1}{r} \rceil$ (and, for $n = 2$, also $|I| \le 3d_1 - 1$). Now we can finally state our formula.

Theorem 3.1. *Notations as above, assume E satisfies conditions AB and T and set $b = c_2(E) \in \mathbb{Z}$. Then the weighted number of jumping rational curves meeting $A_1, ..., A_k$ is given by*

$$J_{d,E}(a.) = \tag{3.15}$$

$$r(g-1) - \frac{1}{2}((a^2 - 2b)L^2 + rD^2) + \frac{1}{2}\big[(-r)(2g - 2 - m_1)$$

$$+ aR_1.L + 2rs_1.D - \sum_{i=2}^{k} t_i R_1.s_i)\big] - h(t., a.)$$

proof. We apply the Riemann-Roch formula in Grothendieck's form ([F],15.2) (though the Hirzebruch form would have worked too) to the vector bundle G and the map $\pi : X \to B$. Clearly $\pi_*(G) = 0$ while $R^1\pi_*(G)$ is a torsion sheaf supported firstly on those $b \in B$ corresponding to irreducible jumping curves C, where it has length $h^1(G_{F_b})$, and secondly on those b corresponding to reducible fibres $F_b = C_1 \cup_p C_2$ such that $h^1(G_{F_b}) \neq 0$ where its length is again equal to this h^1. By property T, E is almost balanced on all reducible fibres F_b. This implies easily that F_b has at most a unique component, say C_1 of degree d_1, such that $h^1(G_{C_1}) \neq 0$, and in this case

$$h^0(G_{C_1}) = 0$$

$$h^1(G_{C_2}(-p)) = 0$$

hence

$$h^1(G_{F_b}) = h^1(G_{C_1}).$$

Now we have

$$E_{C_1} \simeq s\mathcal{O}(j) \oplus (r-s)\mathcal{O}(j-1)$$

with $j = \lceil \frac{ad_1}{r} \rceil$, and it is immediate from this that $h^1(G_{C_1}) \neq 0$ only if $t_I > j$, in which case

$$h^1(G_{F_b}) = h^1(G_{C_1}) = rt_I - ad_1 - r.$$

It follows that the total h^1 coming from reducible fibres equals $h(t., a.)$, so one side of GRR yields $-J_{d,E}(a.) - h(t., a.)$.

Now the other side of GRR generally equals

$$(r1_X + c_1(G) + \frac{1}{2}(c_1^2 - 2c_2)(G))(1_X - \frac{1}{2}K_X + \chi(\mathcal{O}_X)[pt])_2 \tag{3.16}$$

where $[pt]$ is a point and $_2$ denotes degree-2 part. Clearly

$$\chi(\mathcal{O}_X) = 1 - g.$$

Next, the canonical class K_X was computed in [R2][R3] as

$$K_X = -2s_i + (2g - 2 - m_i)F + R_i$$

for any i (we take $i = 1$), where F is a fibre. Given this, the computation of (3.16) is routine, yielding the formula (3.15). $\square$

Example 3.2. Take $n = 2, d = 4$ and let E be the tangent bundle of $\mathbb{P}^2$, with Chern classes $a = 3, b = 3$. It is known that in this case $N_4 = 620, g = 725$. It is easy to compute that

$$m_1 = 284, R_1.L = 5220.$$

We take $D = 7s_1$ and compute that in this case the h^1 contribution from the reducible curves is

$$h(t.a.) = 9180,$$

therefore finally

$$J_{4,T_{\mathbb{P}^2}} = 7944.$$

It is easy to see that on an irreducible rational quartic the tangent bundle cannot have $\mathcal{O}(4)$ as a direct summand so the jumping quartics all have splitting type $(7,5)$. It is also easy to see that a conic-pair $C_1 \cup_p C_2$ with C_1, C_2 irreducible is jumping iff C_1 and C_2 are tangent at p.

Example 3.3. Again for $n = 2$, let E be a semistable bundle with $c_1 = 0, c_2 = m - 1$ corresponding to m general points in $\mathbb{P}^2$ (cf. [OSS]). Then we compute

$$J_{d,E} = (m-1)N_d.$$

For $m = 2, \mathcal{J}_{d,E}$ is just a hyperplane section of V_d corresponding to a certain point in $\mathbb{P}^2$. For $m = 3$, we get an interesting class of quadric sections of V_d.

Remark 3.4. Note that Theorem 3.1 may be applied to the restriction of E to a general linear subspace $A_0 \subset \mathbb{P}^n$. Hence the Theorem generalizes immediately to the case where one of the incidence conditions on the rational curve becomes containment in A_0.

REFERENCES

[F] W. Fulton: 'Intersection theory' Springer 1984.

[G] A. Givental: 'Problems in quantum K-theory', talk at Southern California Algebraic Geometry Seminar, 2/00.

[K] J.Kollár: 'Rational curves on algebraic varieties' Springer 1996.

[OSS] Ch. Okonek, M. Schneider, H. Spindler: 'Vector bundles on projective spaces' Birkhäuser 1980.

[P] R. Pandharipande: 'The canonical class of $M_{0,n}(\mathbb{P}^r, d)$ and enumerative geometry' IMRN 1997, no.4, 173-186.

[R1] Z. Ran: 'Bend, break and count' Isr. J. Math 111 (1999), 109-124.

[R2] Z. Ran: 'Bend, break and count II' (Math. Proc. Camb. Phil. Soc., to appear)

[R3] Z. Ran: 'On the variety of rational space curves' Isr. J. Math (to appear)

[S] E. Sernesi: ' On the existence of certain families of curves' Invent. math. 75 (1984), 25-57.